Optimality Bounds and Optimization for Image Partitioning

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Efficient Algorithms for Global Optimisation Methods
Dagstuhl, November 21, 2011
Motivation – Problem

- Labeling problem:

- Partition image domain $\Omega$ into $L$ regions
- *Discrete* decision at each point in *continuous* domain $\Omega$

Variational Approach:

$$\min_{\ell} \int_{\Omega} s(\ell(x), x) dx + J(\ell)$$

- Local data fidelity
- Regularizer

$\ell(x) = 2$

$\{1, \ldots, L\}$
Motivation – Multiclass Labeling

- **Applications:** Denoising, segmentation, 3D reconstruction, depth from stereo, inpainting, photo montage, optical flow,...

- Spatially continuous formulation avoids metricalation artifacts:
Model – Multi-Class Labeling

▶ Multi-class relaxation: [Lie et al. 06, Zach et al. 08, Lellmann et al. 09, Pock et al. 09]

▶ Embed labels into $\mathbb{R}^L$ as $\mathcal{E} := \{e^1, \ldots, e^L\}$, relax to the unit simplex:

$$
\Delta_L := \{x \in \mathbb{R}^L | x \geq 0, \sum_i x_i = 1\} = \text{conv } \mathcal{E},
$$

$$
\min_{u \in \text{BV}(\Omega, \Delta_L)} f(u), \quad f(u) := \int_{\Omega} \langle u(x), s(x) \rangle dx + \int_{\Omega} \Psi(Du)
$$

▶ Advantages: No explicit parametrization, rotation invariance, convex
Model – Envelope Relaxation

- \( J(\ell) \): Weight boundary length by interaction potential \( d(i,j) \)

- \( J(u) \) implicitly defined as local envelope for given \( d \)

\[ J(u) := \sup_{v \in \mathcal{D}} \int_{\Omega} \langle Du, v \rangle = \int_{\Omega} \underbrace{\sigma_{\mathcal{D}_{\text{loc}}}(Du)}_{\Psi(Du)} \]

\[ \mathcal{D} := \{ v \in (C^\infty_c)^{d \times L} | v(x) \in \mathcal{D}_{\text{loc}} \ \forall x \in \Omega \} , \]

\[ \mathcal{D}_{\text{loc}} := \{ (v^1, \ldots, v^L) \in \mathbb{R}^{d \times L} | \| v^i - v^j \| \leq d(i,j) \ \forall i,j \} . \]
Fractional solutions may occur:

Goal: Find rounding scheme $u^* \mapsto \bar{u}^* : \Omega \to \{e^1, \ldots, e^L\}$ such that

$$f(\bar{u}^*) \leq Cf(u^*_E).$$

for some $C \geq 1$. 

J. Lellmann – Optimality Bounds and Optimization for Image Partitioning
Two-class case: Generalized coarea formula [Strang83, ChanEsedogluNikolova06, Zach et al. 09, Olsson et al. 09]

\[ f(u) = \int_0^1 f(\bar{u}_\alpha) d\alpha, \quad \bar{u}_\alpha := \begin{cases} e^1, & u_1(x) > \alpha, \\ e^2, & u_1(x) \leq \alpha. \end{cases} \]

Also: Choquet integral, Lovász extension, levelable function,…

Consequence: \( C = 1 \), global integral minimizer for a.e. \( \alpha \):

Multi-class generalization (approximate generalized coarea formula):

\[ Cf(u) \geq \int_{\Gamma} f(\bar{u}_\gamma) d\mu(\gamma) = \mathbb{E}_\gamma f(\bar{u}_\gamma) \]

Parameter space: sequences \( \gamma \in \Gamma := (\{1, \ldots, L\} \times [0, 1])^N \)
Optimality – Example
Optimality – Example
Optimality – Example
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Optimality – Example
Theorem (Optimality \cite{LellmannLenzenSchnoerr2011})

Let \( u \in \text{BV}(\Omega, \Delta_L) \), \( s \in L^\infty(\Omega)^L \), \( s \geq 0 \), \( d \) metric. Then

\[
E f(\bar{u}) \leq 2 \frac{\max_{i \neq j} d(i, j)}{\min_{i \neq j} d(i, j)} f(u) \quad \text{and} \quad E f(\bar{u}^*) \leq 2 \frac{\max_{i \neq j} d(i, j)}{\min_{i \neq j} d(i, j)} f(u^*_c).
\]

- Provides “approximate” generalized coarea formula
- Compatible with bounds for finite-dimensional multiway cut, \( \alpha \)-expansion, LP relaxation \cite{Dahlhaus et al. 94, KleinbergTardos 99, Boykov et al. 01, KomodakisTziritas 07}
- Formulated in BV, independent of discretization, true \textit{a priori} bound
Reducing Metrics

- Tight regularizer, but:

\[
\mathcal{D}_{\text{loc}} := \{(v^1, \ldots, v^L) \in \mathbb{R}^{d \times L} \mid \|v^i - v^j\| \leq d(i, j) \ \forall i, j\}.
\]

- Many constraints: \(O(L^2)\)

- Assume there is \((i, j, k)\) such that \(d(i, k) \geq d(i, j) + d(j, k)\). Then

\[
\|v^i - v^k\| \leq \|v^i - v^j\| + \|v^j - v^k\| \leq d(i, j) + d(j, k) \leq d(i, k).
\]

\(\Rightarrow\) Removing constraint for \((i, k)\) does not change \(\mathcal{D}_{\text{loc}}\).

- How to continue?
Reducing Metrics

- Can show:
  - No need for sequential removal
  - Remaining set of constraints is unique
  - Does not change dual constraint set $\mathcal{D}_{\text{loc}}$/regularizer $\psi$

### Algorithm 1 (Reducing Metrics)

- For all $(i, j, k) \in \{1, \ldots, L\}^3$:
  - If $d(i, k) \geq d(i, j) + d(j, k)$: remove constraint $\|v^i - v^k\| \leq d(i, k)$
Reducing Metrics – Uniform

- **Uniform (Potts) metric:** \( d(i, j) = 1_{i \neq j} \)

- No reduction possible
Reducing Metrics – Uniform and Linear

- **Linear metric:** \(d(i, j) = |i - j|\)

- **Full reduction:** \(O(n^2) \rightarrow O(n)\)
Reducing Metrics – Trees

- Tree metric: $d(i, j) = \text{shortest\_path}(T, i, j)$

<table>
<thead>
<tr>
<th>original</th>
<th>reduced</th>
<th>completion</th>
</tr>
</thead>
<tbody>
<tr>
<td>[2x2 matrix]</td>
<td>[2x2 matrix]</td>
<td>[2x2 matrix]</td>
</tr>
</tbody>
</table>

- Full reduction: $O(n^2) \rightarrow O(n)$
Reducing Metrics – Cyclic

- General graph: \( d(i, j) = \text{shortest\_path}(G, i, j) \)
- Continuous analogue: \( \text{TV}_{S^1} \) for angular/orientation data

[StrekalovskiyCremers2011]

- Full reduction: \( O(n^2) \rightarrow O(n) \)
Reducing Metrics – Multiple Components

- Labels quantize vector-valued quantities, e.g. optical flow
- $n$ quantization steps per component
  $\Rightarrow n^2$ labels, $O(n^4)$ constraints!
- Separable metric (linear):
  $$d(((i_1, i_2), (j_1, j_2)) = |i_1 - j_1| + |i_2 - j_2|$$
- Possible: two indicator functions
  - Nonconvex dataterm, needs relaxation $\Rightarrow$ less tight  [GoldlueckeCremers2010]
  - or additional dual variables  [StrekalovskiyGoldlueckeCremers2011]
Reducing Metrics – Multiple Components

- **Separable metric (linear):**

\[ d((i_1, i_2), (j_1, j_2)) = |i_1 - j_1| + |i_2 - j_2| \]

- **Full reduction:** \( O(n^4) \rightarrow O(n^2) \) (\( n^2 \) labels)
Reducing Metrics – Multiple Components

- Separable metric (*uniform*/Potts):

\[ d((i_1, i_2), (j_1, j_2)) = 1_{i_1 \neq j_1} + 1_{i_2 \neq j_2} \]

- Reduction: \( O(n^4) \rightarrow O(n^3) \) (still \( n^2 \) labels)
Setting:
- Tight convex relaxation of multiclass image labeling

Bounds:
- \textit{Probabilistic a priori} bound
- Approximate generalized coarea formula
- Compatible with finite-dimensional results

Reducing Metrics:
- Automatically remove redundant constraints
- Reduces complexity for widely used metrics
- Easy integration into other approaches
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Optimality – Termination

Theorem (Termination)

Let $u \in BV(\Omega, \Delta_L)$. Then (almost surely) Alg. 1 generates a sequence that becomes stationary in some $\bar{u} \in BV(\Omega, \mathcal{E})$.

- Result is in BV
- Independent of data term
Experiments – Iterations
Experiments I

[Images of different color partitions and images]
Experiments II
## Experiments – Results

<table>
<thead>
<tr>
<th>problem</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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<td></td>
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<td></td>
<td></td>
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<tr>
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</table>
Algorithm 1 (Randomized Rounding in BV)

1. **Input:** \( u^0 \in \text{BV}(\Omega, \Delta_L) \)
2. **For** \( k = 1, 2, \ldots \)
3. Sample \( \gamma^k = (i^k, \alpha^k) \in \{1, \ldots, L\} \times [0, 1] \) uniformly
4. \( u^k \leftarrow e^{i^k} \mathbb{1}_{\{u_{ik}^{k-1} > \alpha^k\}} + u^{k-1} \mathbb{1}_{\{u_{ik}^{k-1} \leq \alpha^k\}} \)
5. **Output:** Limit \( \bar{u} \) of \( (u^k) \)

- Parameter space: \( \text{sequences} \ \gamma \in \Gamma := (\{1, \ldots, L\} \times [0, 1])^\mathbb{N} \)
Definition

For some sequence \((\gamma^k)\), if \((u^k_\gamma)\) becomes stationary at some \(u^k_\gamma' \in \mathbb{N}\), denote output \(\bar{u}_\gamma := u^k_\gamma'\). For some functional \(f : BV(\Omega)^L \to \mathbb{R}\), define

\[
\begin{align*}
  f(\bar{u}_\gamma) : \Gamma^\mathbb{N} &\to \mathbb{R} \cup \{+\infty\} \\
  \gamma \in \Gamma^\mathbb{N} &\mapsto f(\bar{u}_\gamma) := \begin{cases} 
  f(u^k_\gamma'), & (u^k_\gamma) \text{ stationary at } u^k_\gamma' \in BV(\Omega)^L, \\
  +\infty, & \text{otherwise.}
  \end{cases}
\end{align*}
\]

- Can show: \(\mathbb{P}_\gamma(f(\bar{u}_\gamma) < \infty) = 1\).
  - Generates “output” in finite time almost surely
  - Output is in \(BV(\Omega)^L\) almost surely
Reducing Metrics – Uniform and Linear

- **Truncated linear metric:** \( d(i, j) = \min\{3, |i - j|\} \)

- Half the number of constraints