Overview
Continuous Multi-Class Image Labeling
Relaxation yields linear data term
Use weighted variant of total variation on vector fields as regularizer
Allows for non-trivial interaction potentials, can be optimized quickly
Models Euclidean distance interaction potentials exactly, non-Euclidean distances can be approximated
Globally optimized using Nesterov’s first-order approach with explicit optimality bounds

Continuous Problem Formulation
Problem
For each pixel \(x \in \Omega \subseteq \mathbb{R}^2\), find a class label \(\ell(x) \in \{1, \ldots, L\}\) according to local data fidelity and regularization term
\[ J = \text{Combinatorial problem.} \]

Relaxation
Identify the \(s\)-th label with \(\ell' \) and relax to the unit simplex in \(\mathbb{R}^L\), find
\[ \min_{u \in \mathbb{R}^L} \int_{\Omega} \langle u(x), x(x) \rangle dx + J(u), \]
where \( C = \{ u : \Omega \rightarrow \mathbb{R} \mid u(x) \geq 0, \sum_{\ell=1}^{L} u(x) = 1 \} \) and \( J \) penalizes label changes.
\( \Rightarrow \) Convex problem for any data term. Formulating \( J \) means trading off generality vs. simplicity of computation.

Total Variation Based Regularizer
Approach
Fix an interface potential \(d : \{1, \ldots, L\} \rightarrow \mathbb{R} \) and require regularization according to boundary length with weight depending on labels \(i \) and \(j \) of adjoining regions,
\[ J(i' \downarrow s + i' 1_{j \downarrow s}) = d(i, j) \text{Per}(S) \]
for any set \( S \subset \Omega \) with finite perimeter. If \( J \) convex, positively homogeneous and \( J(u) = 0 \) for constant \( u \), then \( d \) must be a metric.
Euclidean Distances
Idea: Use linear modification of total variation on vectors,
\[ J(u) = \text{TV}(u) = \int_{\Omega} ||D(u)||_F dx, \]
here \( ||D(\cdot)||_F \) Frobenius norm of the Jacobian and \( A \in \mathbb{R}^{k \times d} \). Then
\[ \text{TV}(i' \downarrow s + i' 1_{j \downarrow s}) = ||u' - u||_F \text{Per}(S). \]
If \( d(i, j) = ||a' - a|| \), i.e. \( a \) is an Euclidean distance, \( J \) is exact for hard labeling.

Approximation of Non-Euclidean Distances
In case \( d \) is non-Euclidean, e.g. \( d(i, j) = \min \{1, |i - j|\} \): Set \( D_h = d(i, j)^2 \) and compute Euclidean approximation by minimizing
\[ ||E - D_h||_F^2 \]
over all Euclidean distance matrices \( E \) by solving a convex semidefinite program. Example: truncated linear distance, absolute error bound \( E_E = 0.145: \)
\[
\begin{pmatrix}
0 & 1 & 2 & 1 & 2 & 0
1 & 0 & 1 & 2 & 0 & 1
2 & 1 & 0 & 1 & 2 & 0
1 & 2 & 0 & 1 & 2 & 0
2 & 0 & 1 & 2 & 0 & 1
1 & 2 & 0 & 1 & 2 & 0
\end{pmatrix}
\]

Discretized Problem
Use forward differences and support function representation of discrete total variation to get a bilinear saddle point problem:
\[ \min_{u \in C} \langle u, s \rangle + \langle (L(x), v) - (b, v) \rangle, \]
where \( L \) discretization of gradient and
\[ C = \{ u \in \mathbb{R}^n | u_i \in \Delta_i, i = 1, \ldots, n \}, \]
\[ D = \{ p \in \mathbb{R}^p \mid ||p||_2 \leq 1 \} \subseteq \mathbb{R}^{n \times n}. \]
Existence and strong duality follow from boundedness of \( C, D \). Projections onto \( C, D \) can be computed exactly in a finite number of steps.

Optimization
Solve non-smooth problem using Nesterov’s approach [1]
First-order method: exploit sparsity
Combines controlled smoothing with accumulated gradients
Requires only evaluations of \( L \) and projections onto \( C, D \)
\( O(1/n) \) convergence, compared to \( O(1/\sqrt{n}) \) for subgradient methods
Explicit suboptimality bound for given number of iterations, \( O(1/n \sqrt{T \langle \Omega \}}) \) to find \( \varepsilon \)-optimal solution
• Fully automatic, parameter-free

Experiments
Figure: Convexity for stereo disparity estimation with non-Euclidean distance and 16 disparities:
Objective vs. number of iterations,
Input, proposed method with varying distance: Input, proposed method with non-binary solution; after binarization the result is in accordance with the expected solution.
Figure: Simultaneous segmentation and background reconstruction: Noisy image; background and foreground reconstructed using a non-uniform distance.

References
Smooth minimization of non-smooth functions.
A convex approach for computing minimal partitions.

Algorithm 1 Convex Multi-Class Labeling
\[ \begin{align*}
\text{Input:} & \quad s, C, x, F, a, b, D, \varepsilon, \delta, T, N \in \mathbb{N}. \\
\text{Output:} & \quad u \in \mathbb{R}^n, v \in \mathbb{R}^m, \pi \in \mathbb{R}^m. \\
\text{Let} & \quad u = \frac{a}{\varepsilon}, v = \frac{b}{\varepsilon}, \pi = D, \delta. \\
\text{Set} & \quad G = \mathbb{R}^n = \mathbb{R}^m = 0. \\
\text{for} & \quad k = 0, \ldots, N \text{ do} \\
& \quad V = V + \frac{1}{k+1} (G(x)^{\delta} - \delta x) \\
& \quad u = u - \frac{1}{k+1} (\nabla x^T V) \\
& \quad v = v - \frac{1}{k+1} (\nabla x^T V) \\
& \quad (x, D) = \frac{1}{k+1} (x^T + \varepsilon^T) + \frac{1}{k+1} (\frac{1}{k+1} D^T + \pi^T) \\
\text{end for}
\end{align*} \]