Precise Relaxation for Motion Estimation

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Variational motion estimation

\[
\min_{u: \Omega \to \mathbb{R}^2} \int_{\Omega} D(I^1_\sigma(x), I^2_\sigma(x + u)) \, dx + \lambda \int_{\Omega} d \| Du \|
\]

\( u \) scalar (disparity, stereo) or vector field (optical flow, image registration)

Images: Lellmann, Strekalovskiy, Kötter, Cremers’13
Variational methods

We reconstruct the unknown data $u$ from the measurements $b$ by minimizing the energy

$$\min_u \{ D(T(u); b) + R(u) \}$$

Intuitive (what do we want) and modular (reusable) In practice: often

$$\min_{u: \Omega \rightarrow X} \int_{\Omega} \rho(x, u(x)) dx + \lambda \int_{\Omega} \sigma(\nabla u) dx$$
Convexity assures that every local minimizer of the energy is also a global minimizer.
Variational motion estimation

\[
\min_{u: \Omega \to \mathbb{R}^2} \int_{\Omega} D(I_1^1(x), I_2^2(x + u)) \, dx + \lambda \int_{\Omega} d \| Du \|
\]

\(u\) scalar (disparity, stereo) or vector field (optical flow, image registration)

Images: Lellmann, Strekalovskiy, Kötter, Cremers’13
Non-convexity

- Real-world disparity estimation/depth from stereo:

- Data-dependent nonconvexity at every point
Exact convex relaxation
Exact convex relaxation
Exact convex relaxation
Exact convex relaxation
Drawback

This is generally as hard as minimizing the original energy!
Objectives are often sums

$$\min_{u: \Omega \to X} \int_{\Omega} \rho(x, u(x)) \, dx + \lambda \int_{\Omega} \sigma(\nabla u) \, dx$$
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\[ f = f_1 + f_2 \]
Separate exact relaxation
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Lifting

Hard decisions are replaced by soft "probabilities"
Lifting

We extend the problem

$$\min_{u' : \Omega \to X} f'(u')$$

to the probability measures:

$$\min_{u : \Omega \to \mathcal{P}(X)} f(u)$$

The new energy $f$ should agree with $f'$ on Dirac measures, and not create artificial minimizers.

Potts ‘52; Boykov et al. ‘98, ‘01; Kleinberg, Tardos ‘01
Zach et al. ‘08; Lellmann, Becker, Schnörr ‘09; Chambolle, Cremers, Pock ‘11; Yuan, Bae, Tai, Boykov’10
Young measures: Young ‘37; Currents: Schwartz ‘51; de Rham ‘55; Federer ‘69
Paired calibrations: Brakke ‘91; Alberti, Bouchitté, Dal Maso ‘01
Linear relaxation

- Lifting + relaxation using the biconjugate:

\[
\int_{\Omega} \rho(u(x)) \, dx \quad \rightsquigarrow \quad \int_{\Omega} \rho^{**}(u(x)) \, dx
\]
Linear relaxation

• Lifting + relaxation using the biconjugate:

\[ \int_{\Omega} \rho(u(x)) dx \Leftrightarrow \int_{\Omega} \rho^{**}(u(x)) dx \]

• Linear relaxation (1-sparse solutions):

\[ \rho(z) = \begin{cases} 
\rho(t^i), & z = e^i, \ i \in \{1, \ldots, L\}, \\
+\infty, & \text{otherwise}.
\end{cases} \]
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  \[ \min_{u \in \text{BV}(\Omega, \mathcal{P}(X))} f(u) := \int_\Omega \langle u(x), s(x) \rangle \, dx + \int_\Omega d\Psi(Du) \]
Lifting + linear relaxation
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\[ f_1^\# \]
\[ f_2^\# \]
\[ f^\# = f_1^\# + f_2^\# \]
Lifting + linear relaxation
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Lifting + linear relaxation
RGB-depth segmentation
(Diebold et al., SSVM ’15)

3D reconstruction
(Kolev et al., Int. J. Comp. Vis. ’09)

Restoring manifold-valued data
(Cremers, Strekalovskiy ’12,
Lellmann et al., ICCV’13)
related: Weinmann, Demart, Storath’14;
Bergman et al.’14
Label bias

The solution tends strongly towards the chosen labels!
Lifting + linear relaxation
Precise relaxation

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- New: Precise relaxation (2-sparse!)
  \[ \rho(z) = \begin{cases} 
    \rho((1 - \alpha)t^i + \alpha t^{i+1}), & z = (1 - \alpha)e^i + \alpha e^{i+1}, \\
    +\infty, & \text{otherwise.}
  \end{cases} \]

- Linear programs: solution lies on vertex.
- Here: solution between two vertices

Möllenhoff, Laude, Möller, Lellmann, Cremers, CVPR’16
related: Zach, Kohli, ECCV’12; Zach, AISTATS’13
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Relaxing the regularizer

**Proposition 4.** The convex envelope of (15) is

\[ \Phi^{**}(g) = \sup_{q \in \mathcal{K}} \langle q, g \rangle, \]  

(17)

where \( \mathcal{K} \subset \mathbb{R}^{k \times d} \) is given as:

\[ \mathcal{K} = \left\{ q \in \mathbb{R}^{k \times d} \mid \right. \]

\[ \left. \left| q^T (1^\alpha_i - 1^\beta_j) \right|_2 \leq \left| \gamma_i^\alpha - \gamma_j^\beta \right|, \right\} \]

\( \forall 1 \leq i \leq j \leq k, \forall \alpha, \beta \in [0, 1] \).  

\[ \]  

**Proposition 5.** In case the labels are ordered, i.e., \( \gamma_1 < \gamma_2 < \ldots < \gamma_L \), then the constraint set \( \mathcal{K} \) from Eq. (36) is equal to

\[ \mathcal{K} = \left\{ q \in \mathbb{R}^{k \times d} \mid |q_i|_2 \leq \gamma_{i+1} - \gamma_i, \forall i \right\}. \]  

(19)
New: lifting + precise relaxation
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New: lifting + precise relaxation
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Results – disparity estimation
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linear lifting

$L = 2$
Results – disparity estimation

linear lifting

$L = 4$
Results – disparity estimation

linear lifting

$L = 8$
Results – disparity estimation

\[ L = 16 \]

Möllenhoff, Laude, Möller, Lellmann, Cremers, CVPR’16
Results – disparity estimation

linear lifting

$L = 32$

Möllenhoff, Laude, Möller, Lellmann, Cremers, CVPR’16
Results – disparity estimation

\[ L = 32 \]

linear lifting

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precise lifting

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Results – disparity estimation

\[ L = 32 \]

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Results – disparity estimation

\[ L = 270 \]  
linear lifting

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precise lifting

Möllenhoff, Laude, Möller, Lellmann, Cremers, CVPR’16
Related work

- Zach, Kohli’12; Zach’13; Fix, Agarwal’14
  - piecewise convex
  - different relaxation
  - MRF-based – not isotropic yet
Outlook – multiple dimensions
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Large displacement optical flow

- Input sequence: (Middlebury)
- LP-style relaxation (L. et al.'13):
  - 7x7, 5.2GB, 33min, aep 2.65
- Ours, 2x2 labels:
  - 0.63GB, 17min, aep 1.28
- Ground truth:
- Product spaces (Goldluecke et al.'13):
  - 28x28, 9.3GB, 60min, aep 1.39
- Ours, 6x6 labels:
  - 10.1GB, 56min, aep 0.9
Large displacement optical flow

References:

Take-home

- Goal: global minimizer of nonconvex energies
- Lift into larger space
- Relax piecewise convex
- Much smaller problems, often 2-4 labels enough
Relaxation Methods in Variational Image Processing

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