Variational problems with finite and infinite label spaces

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Variational problems
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The problem

Given data $f$, find the image information $u$ that solves

$$f = T(u) + n,$$

where $T$ models the relation between $u$ and $f$, and $n$ is noise: Minimise

$$\min_{u: \Omega \to Y} \left\{ D(T(u); f) + J(u) \right\}$$

data-/fidelity term, compatibility with measurements $f$

regulariser, prior knowledge (specific to problem)
Finite label spaces
Labelling problems

- In many interesting problems the range is discrete: A discrete decision is required in every $x \in \Omega$. 

with: V. Corona, C. Schönlieb, J. Acosta-Cabronero, P. Nestor/DZNE Magdeburg
Labelling problems

Variational formulation:

\[
\min_{u: \Omega \rightarrow \{1, \ldots, L\}} \int_{\Omega} s(u(x), x) dx + J(u)
\]

Local data term + regulariser

Combinatorial/geometrical constraints → nonconvex!
Relaxation

- Replace the function $u : \Omega \to Y$ by the map $u' : \Omega \to \mathcal{M}(Y)$ from $\Omega$ into the set of **Dirac/point measures** on $Y$ (vector space!):

  $$u(x) = y \iff u'(x) = \delta_y.$$ 

- Relax to the set of all maps $u' : \Omega \to \mathcal{P}(Y)$ into the set of **probability measures** on $Y$ – always convex!

Relaxation – finite labels

- **Finite case** $Y = \{1, \ldots, L\}$: [Lie et al. 06, Zach et al. 08, Lellmann et al. 09, Pock et al. 09]

  $$y = \{1, \ldots, L\}$$

- Probability measures parametrised by unit simplex in $\mathbb{R}^L$:

  $$\mathbb{P}(Y) := \{x \in \mathbb{R}^L | x \geq 0, \sum x_i = 1\},$$

  $$\min_{u' \in BV(\Omega, \mathbb{P}(Y))} f(u'), \quad f(u') := \int_\Omega \langle u'(x), s(x) \rangle dx + J(u')$$

- **Convex** problem. Question: How to formulate/extend regulariser?
Relaxation – regulariser

- **Length-based regularisation**: Boundary length weighted by interaction potential (metric) \(d(i, j)\) depending on the labels:

- \(J(u)\) implicitly defined by (local) **convex envelope**:

\[
J(u) := \sup_{v \in \mathcal{D}} \int_{\Omega} \langle u', \text{Div} \, v \rangle \, dx = \int_{\Omega} \sigma_{\mathcal{D}_{\text{loc}}}(Du'), \\
\Psi(Du'),
\]

\[
\mathcal{D} := \{ v \in (C^\infty_c)^{d\times L} | v(x) \in \mathcal{D}_{\text{loc}} \quad \forall x \in \Omega \}, \\
\mathcal{D}_{\text{loc}} := \{ (v^1, \ldots, v^L) \in \mathbb{R}^{d\times L} | \| v^i - v^j \| \leq d(i, j) \quad \forall i, j \}.
\]

- Overall **convex, non-smooth** problem
Example
Infinite label spaces
Manifold-valued problems

- Interferometric data (InSAR) contains only the phase \((d \mod 2\pi)\) of the distance:
The range of the data $u$ is constrained to a (Riemannian) manifold $\mathcal{M}$. Generalised Rudin-Osher-Fatemi:

$$\min_{u: \Omega \to \mathcal{M}} \int_{\Omega} d_{\mathcal{M}}(u(x), f(x))^2 \, dx + J(u),$$

$$\approx \int_{\Omega} \|Du\| \, dx$$

Total Variation (TV)-based: $[\text{Giaquinta, Mucci}]$

$$J(u) = TV_{\mathcal{M}}(u) = \int_{\Omega \setminus S_u} |\nabla u| \, dx + \int_{S_u} d_{\mathcal{M}}(u^-, u^+) \, d\mathcal{H}^{m-1} + J_C(u).$$

- Nonconvex due to the constraints!
- Nonconvex optimisation on manifolds:
  - $[\text{Absil, Mahony, Sepulchre’07 – smooth}]$
  - $[\text{WeinmannDemartStorath14, Bergman et al.’14 – nonsmooth}]$
Manifolds – approach

Again $u'(x)$ is a probability measure on $\mathcal{M}$:

$$
\min_{u':\Omega \to \mathcal{P}(\mathcal{M})} \sup_{p: \Omega \times \mathcal{M} \to \mathbb{R}^m} \int_{\Omega} \langle u', s \rangle \, dx + \lambda \int_{\Omega} \langle u', \text{Div} \, p \rangle \, dx
$$

s.t. $\|p(x, z_1) - p(x, z_2)\|_2 \leq d_M(z_1, z_2), \forall z_1, z_2 \in \mathcal{M}, \forall x \in \Omega$

Replace Lipschitz constraint by gradient-based formulation:

$$
\ldots \text{s.t. } \|D_z p(x, \cdot)\|_\sigma \leq 1, \quad \forall z \in \mathcal{M}, \forall x \in \Omega,
$$

$\| \cdot \|_\sigma$ spectral norm.

Convex, nonsmooth, uses manifold structure.
Generalised Rudin-Osher-Fatemi

- $L^2 - TV$ (Rudin-Osher-Fatemi)

$$\min_{u: \Omega \rightarrow \mathcal{M}} D(u) + \lambda TV_{\mathcal{M}}(u)$$

- properties similar to scalar case – contrast loss, jump preservation, …
Application – surface normals
Application – orientations in SO(3)
General convex regularisers

- General convex regularization:

\[
\min_{u: \Omega \rightarrow \mathcal{M}} D(u) + \int_{\Omega \setminus S_u} h(x, \nabla u) \, dx + \lambda \int_{S_u} \min\{\gamma, d_{\mathcal{M}}(u^{-}, u^{+})\} \, d\mathcal{H}^{m-1}.
\]

- Approximation: [cf. StrekalovskiyChambolleCremers2012]

\[
\min_{u': \Omega \rightarrow P(\mathcal{M})} \max_{p: \Omega \times \mathcal{M} \rightarrow \mathbb{R}^m, q: \Omega \times \mathcal{M} \rightarrow \mathbb{R}} \int_{\Omega} \langle u', s \rangle \, dx + \lambda \int_{\Omega} \langle u', \text{Div} \, p - q \rangle \, dx
\]

s.t. \(\|p(x, z_1) - p(x, z_2)\|_2 \leq d_{\mathcal{M}}(z_1, z_2), \forall z_1, z_2 \in \mathcal{M}, \forall x \in \Omega,\)

\(q(x, z) \geq h^*(x, D_z p(x, z)), \forall z \in \mathcal{M}, \forall x \in \Omega,\)
General convex regularisers

- Approximation of Mumford-Shah model on manifolds:

\[
\min_{u: \Omega \to M} D(u) + \lambda \int_{\Omega \setminus S_u} \| \nabla u(x) \|^2 dx + \gamma \mathcal{H}^{m-1}(S_u)
\]
Bregman iteration [Osher, Burger, Goldfarb, Xu, Yin’05] – “convexity splitting”

\[
\min_u D(u) + 0 = \min_u D(u) + \underbrace{J(u) - J(u)}_{\text{convex concave}}
\]

- **Linearise** the concave part:

\[
u^{k+1} = \arg\min_u D(u) + J(u) - (J(u^k) + \langle v^k, u - u^k \rangle), \quad v^k \in \partial J(u^k)
\]

- On scalar data, gradually introduces details and converges to the input → stop if suitable solution found
  - scalar case: \( u(x) \) are real values
  - here: \( u(x) \) are probability distributions
Conclusion

- **Finite label spaces – labelling/segmentation**
  - for *labelling* problems
  - convex relaxation in the function space

- **Infinite label spaces – manifolds**
  - for problems with values in $\mathbb{R}^n, S^n, SO(3), \ldots$
  - also for more general convex regularisers
  - similar properties as in real-valued case: contrast loss, jump preservation, Bregman, \ldots
  - source code available
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Nonconvex data terms

- *negative $L^2$–TV (nonconvex)*

\[
\min_{u: \Omega \rightarrow \mathcal{M}} -D(u) + \lambda \text{TV}_\mathcal{M}(u)
\]
Applications – optical flow