Optimality Bounds and Optimization for Image Partitioning

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Efficient Algorithms for Global Optimisation Methods
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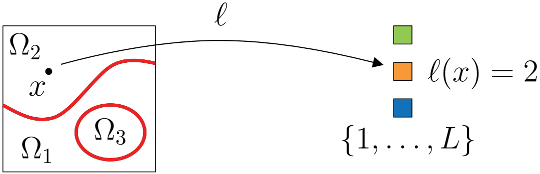
Motivation – Problem

- Labeling problem:

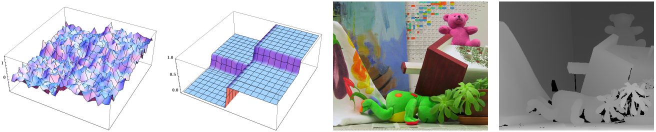


- Partition image domain Ω into L regions
- *Discrete* decision at each point in *continuous* domain Ω
- Variational Approach:

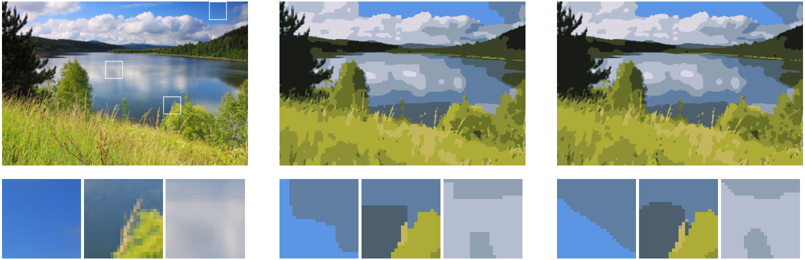
$$\min_{\ell} \underbrace{\int_{\Omega} s(\ell(x), x) dx}_{\text{local data fidelity}} + \underbrace{J(\ell)}_{\text{regularizer}}$$

Motivation – Multiclass Labeling

- **Applications:** Denoising, segmentation, 3D reconstruction, depth from stereo, inpainting, photo montage, optical flow,...



- Spatially continuous formulation avoids metrication artifacts:



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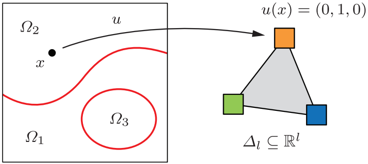
Model – Multi-Class Labeling

- Multi-class relaxation: [Lie et al. 06, Zach et al. 08, Lellmann et al. 09, Pock et al. 09]



- Embed labels into $\mathbb{R}^L$ as $\mathcal{E} := \{e^1, \ldots, e^L\}$, relax to the unit simplex:

$$\Delta_L := \{x \in \mathbb{R}^L | x \geqslant 0, \sum_i x_i = 1\} = \operatorname{conv} \mathcal{E},$$

$$\min_{u \in \mathrm{BV}(\Omega, \Delta_L)} f(u), \quad f(u) := \int_\Omega \langle u(x), s(x) \rangle dx + \int_\Omega \Psi(Du)$$

- Advantages: No explicit parametrization, rotation invariance, *convex*

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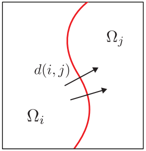
Model – Envelope Relaxation

- $J(\ell)$: Weight boundary length by *interaction potential* $d(i,j)$



- $J(u)$ implicitly defined as local envelope for given d

[ChambolleCremersPock08,LellmannSchnoerr10]

$$
\begin{aligned}
J(u) &:= \sup_{v \in \mathcal{D}} \int_\Omega \langle Du, v \rangle = \int_\Omega \underbrace{\sigma_{\mathcal{D}_{\text{loc}}}(Du)}_{\Psi(Du)}, \\
\mathcal{D} &:= \{ v \in (C_c^\infty)^{d \times L} | v(x) \in \mathcal{D}_{\text{loc}} \; \forall x \in \Omega \}, \\
\mathcal{D}_{\text{loc}} &:= \{ (v^1, \ldots, v^L) \in \mathbb{R}^{d \times L} | \| v^i - v^j \| \leqslant d(i,j) \; \forall i,j \}.
\end{aligned}
$$

- *Fractional* solutions may occur:

- **Goal:** Find *rounding scheme* $u^* \mapsto \bar{u}^* : \Omega \to \{e^1, \ldots, e^L\}$ such that

$$f(\underbrace{\bar{u}^*}_{\text{rounded relaxed solution}}) \leqslant C f(\underbrace{u^*_{\mathcal{E}}}_{\text{best integral solution}}).$$

for some $C \geqslant 1$.

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Rounding – Generalized Coarea Formula

- Two-class case: Generalized *coarea formula* [Strang83, ChanEsedogluNikolova06, Zach et al. 09, Olsson et al. 09]

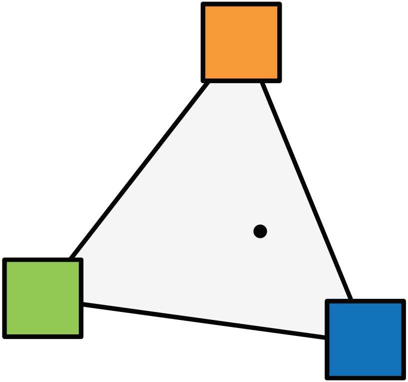
$$f(u) \quad = \quad \int_0^1 f(\bar{u}_\alpha) d\alpha\,, \quad \bar{u}_\alpha := \begin{cases} e^1, & u_1(x) > \alpha, \\ e^2, & u_1(x) \leqslant \alpha. \end{cases}$$

- Also: *Choquet integral, Lovász extension, levelable function*,...
- Consequence: $C = 1$, global *integral* minimizer for a.e. α:
- Multi-class generalization (*approximate* generalized coarea formula):

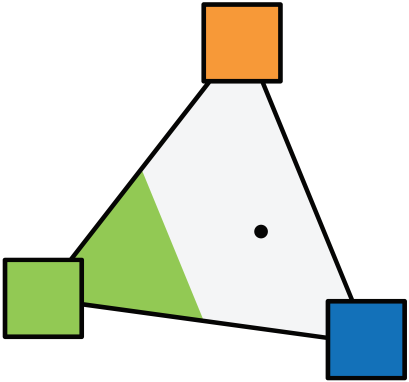
$$C f(u) \quad \geqslant \quad \int_\Gamma f(\bar{u}_\gamma) d\mu(\gamma) = \mathbb{E}_\gamma f(\bar{u}_\gamma)$$

- Parameter space: *sequences* $\gamma \in \Gamma := (\{1, \ldots, L\} \times [0,1])^{\mathbb{N}}$

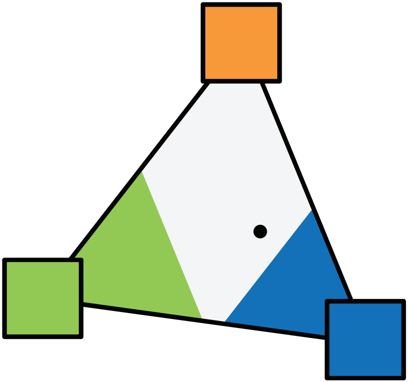
Optimality – Example



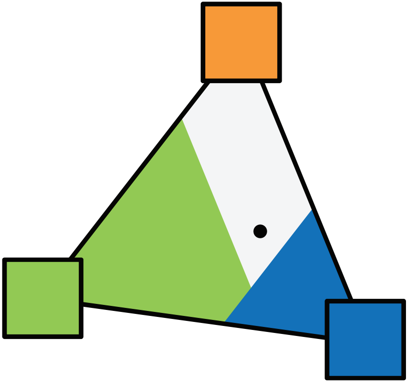
Optimality – Example



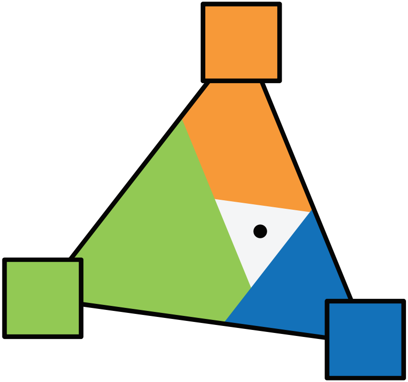
Optimality – Example



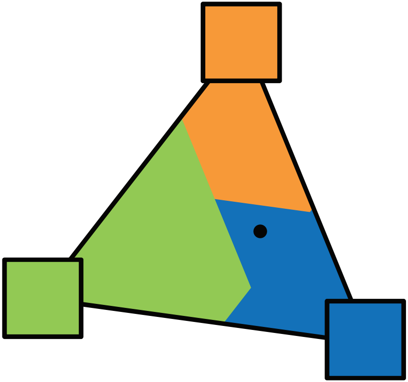
Optimality – Example



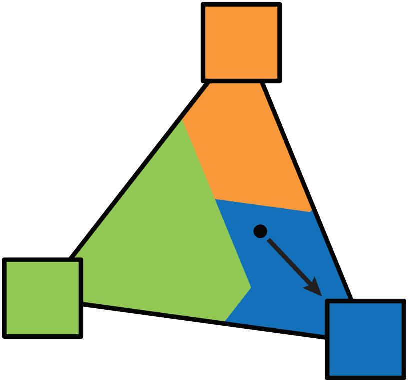
Optimality – Example



Optimality – Example



Optimality – Example



Optimality – Main Result

Theorem (Optimality [LellmannLenzenSchnoerr2011])

Let $u \in \mathrm{BV}(\Omega, \Delta_L)$, $s \in L^\infty(\Omega)^L$, $s \geqslant 0$, d *metric. Then*

$$\mathbb{E}f(\bar{u}) \leqslant 2\frac{\max_{i \neq j} d(i,j)}{\min_{i \neq j} d(i,j)} f(u) \quad \text{and} \quad \mathbb{E}f(\bar{u}^*) \leqslant 2\frac{\max_{i \neq j} d(i,j)}{\min_{i \neq j} d(i,j)} f(u^*_{\mathcal{E}}).$$

- Provides "approximate" generalized coarea formula
- Compatible with bounds for finite-dimensional multiway cut, α-expansion, LP relaxation [Dahlhaus et al. 94, KleinbergTardos 99, Boykov et al. 01, KomodakisTziritas 07]
- Formulated in BV, independent of discretization, true *a priori* bound

Reducing Metrics

- Tight regularizer, but:

$$\mathcal{D}_{\text{loc}} \quad := \quad \{(v^1, \ldots, v^L) \in \mathbb{R}^{d \times L} | \|v^i - v^j\| \leqslant d(i,j) \; \forall i,j\} \, .$$

- Many constraints: $O(L^2)$
- Assume there is (i, j, k) such that $d(i,k) \overset{(=)}{\geqslant} d(i,j) + d(j,k)$. Then

$$\|v^i - v^k\| \leqslant \|v^i - v^j\| + \|v^j - v^k\| \leqslant d(i,j) + d(j,k) \leqslant d(i,k).$$

$\Rightarrow$ Removing constraint for (i,k) *does not change* $\mathcal{D}_{\text{loc}}$.

- How to continue?

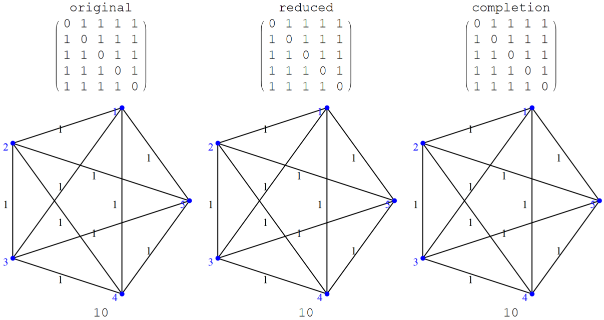
Reducing Metrics

- Can show:
  - No need for sequential removal
  - Remaining set of constraints is unique
  - Does *not* change dual constraint set $\mathcal{D}_{\text{loc}}$/regularizer Ψ

Algorithm 1 (Reducing Metrics)

- For all $(i,j,k) \in \{1,\ldots,L\}^3$:
  - If $d(i,k) \geqslant d(i,j) + d(j,k)$: remove constraint $\|v^i - v^k\| \leqslant d(i,k)$

- Uniform (Potts) metric: $d(i,j) = 1_{i \neq j}$



- No reduction possible

Reducing Metrics – Uniform and Linear

- Linear metric: $d(i,j) = |i-j|$

original

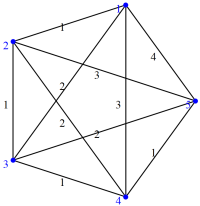
$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 0 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$

reduced

$$\begin{pmatrix} 0 & 1 & \infty & \infty & \infty \\ 1 & 0 & 1 & \infty & \infty \\ \infty & 1 & 0 & 1 & \infty \\ \infty & \infty & 1 & 0 & 1 \\ \infty & \infty & \infty & 1 & 0 \end{pmatrix}$$

completion

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 0 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$



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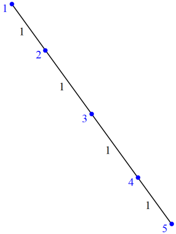

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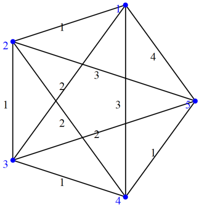

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- Full reduction: $O(n^2) \to O(n)$

Reducing Metrics – Trees

- Tree metric: $d(i,j) = \text{shortest\_path}(T,i,j)$

original

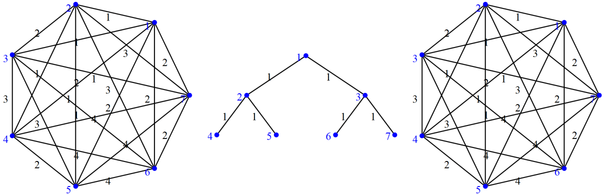
$$\begin{pmatrix} 0 & 1 & 1 & 2 & 2 & 2 & 2 \\ 1 & 0 & 2 & 1 & 1 & 3 & 3 \\ 1 & 2 & 0 & 3 & 3 & 1 & 1 \\ 2 & 1 & 3 & 0 & 2 & 4 & 4 \\ 2 & 1 & 3 & 2 & 0 & 4 & 4 \\ 2 & 3 & 1 & 4 & 4 & 0 & 2 \\ 2 & 3 & 1 & 4 & 4 & 2 & 0 \end{pmatrix}$$

reduced

$$\begin{pmatrix} 0 & 1 & 1 & \infty & \infty & \infty & \infty \\ 1 & 0 & \infty & 1 & 1 & \infty & \infty \\ 1 & \infty & 0 & \infty & \infty & 1 & 1 \\ \infty & 1 & \infty & 0 & \infty & \infty & \infty \\ \infty & 1 & \infty & \infty & 0 & \infty & \infty \\ \infty & \infty & 1 & \infty & \infty & 0 & \infty \\ \infty & \infty & 1 & \infty & \infty & \infty & 0 \end{pmatrix}$$

completion

$$\begin{pmatrix} 0 & 1 & 1 & 2 & 2 & 2 & 2 \\ 1 & 0 & 2 & 1 & 1 & 3 & 3 \\ 1 & 2 & 0 & 3 & 3 & 1 & 1 \\ 2 & 1 & 3 & 0 & 2 & 4 & 4 \\ 2 & 1 & 3 & 2 & 0 & 4 & 4 \\ 2 & 3 & 1 & 4 & 4 & 0 & 2 \\ 2 & 3 & 1 & 4 & 4 & 2 & 0 \end{pmatrix}$$



21 | 6 | 21

- Full reduction: $O(n^2) \to O(n)$

Reducing Metrics – Cyclic

- General graph: $d(i,j) = \text{shortest\_path}(G,i,j)$
- Continuous analogue: $\text{TV}_{\mathcal{S}^1}$ for angular/orientation data

[StrekalovskiyCremers2011]

original

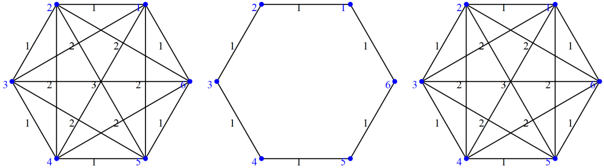
0	1	2	3	2	1
1	0	1	2	3	2
2	1	0	1	2	3
3	2	1	0	1	2
2	3	2	1	0	1
1	2	3	2	1	0

reduced

0	1	∞	∞	∞	1
1	0	1	∞	∞	∞
∞	1	0	1	∞	∞
∞	∞	1	0	1	∞
∞	∞	∞	1	0	1
1	∞	∞	∞	1	0

completion

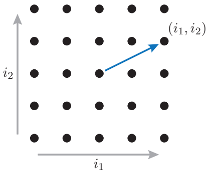
0	1	2	3	2	1
1	0	1	2	3	2
2	1	0	1	2	3
3	2	1	0	1	2
2	3	2	1	0	1
1	2	3	2	1	0



15 | 6 | 15

- Full reduction: $O(n^2) \to O(n)$

Reducing Metrics – Multiple Components



- Labels quantize vector-valued quantities, e.g. optical flow
- n quantization steps per component
  $\Rightarrow n^2$ labels, $O(n^4)$ constraints!
- Separable metric (*linear*):

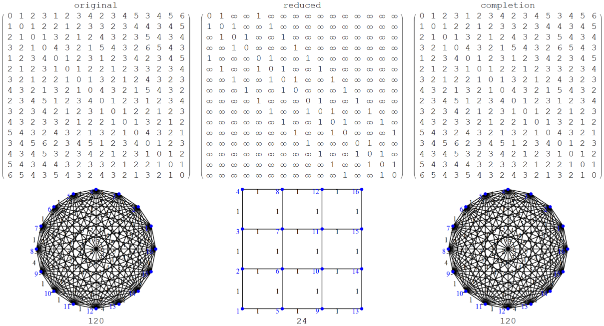
$$d((i_1, i_2), (j_1, j_2)) = |i_1 - j_1| + |i_2 - j_2|$$

- Possible: *two* indicator functions
  - Nonconvex dataterm, needs relaxation $\Rightarrow$ less tight [GoldlueckeCremers2010]
  - or additional dual variables [StrekalovskiyGoldlueckeCremers2011]

Reducing Metrics – Multiple Components

- Separable metric (*linear*):

$$d((i_1, i_2), (j_1, j_2)) = |i_1 - j_1| + |i_2 - j_2|$$

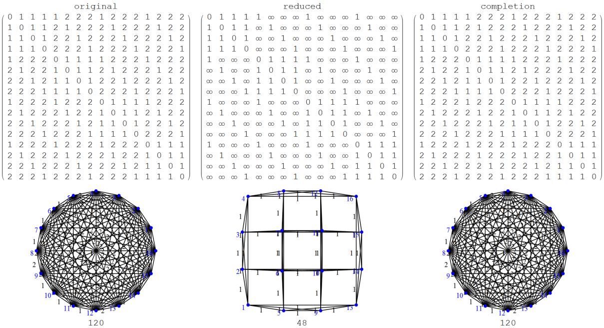


- Full reduction: $O(n^4) \to O(n^2)$ $\quad$ (n^2 labels)

Reducing Metrics – Multiple Components

- Separable metric (*uniform/Potts*):

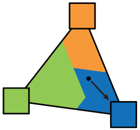
$$d((i_1, i_2), (j_1, j_2)) = 1_{i_1 \neq j_1} + 1_{i_2 \neq j_2}$$



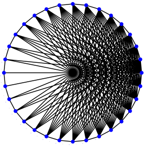
- Reduction: $O(n^4) \to O(n^3)$ (still n^2 labels)

Summary

- **Setting:**
  - Tight convex relaxation of multiclass image labeling
- **Bounds:**
  - *Probabilistic a priori* bound
  - Approximate generalized coarea formula
  - Compatible with finite-dimensional results



- **Reducing Metrics:**
  - Automatically remove redundant constraints
  - Reduces complexity for widely used metrics
  - Easy integration into other approaches



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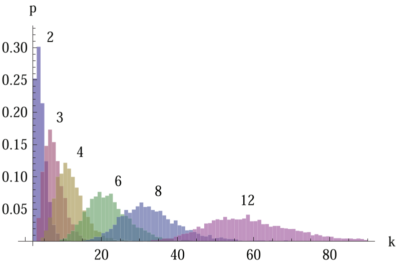
Optimality – Termination

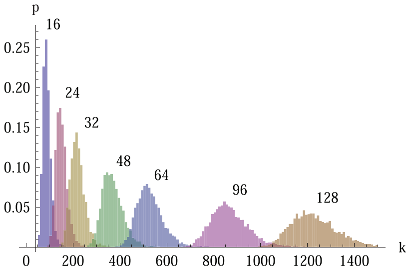
Theorem (Termination)

Let $u \in \mathrm{BV}(\Omega, \Delta_L)$. Then (almost surely) Alg. 1 generates a sequence that becomes stationary in some $\bar{u} \in \mathrm{BV}(\Omega, \mathcal{E})$.

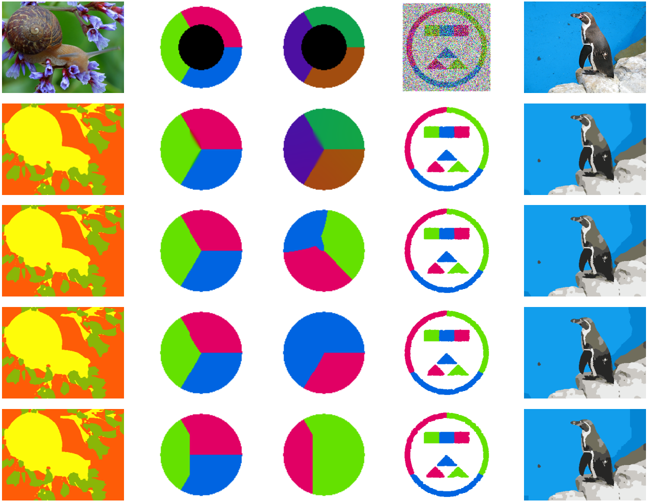
- Result is in BV
- Independent of data term

Experiments – Iterations

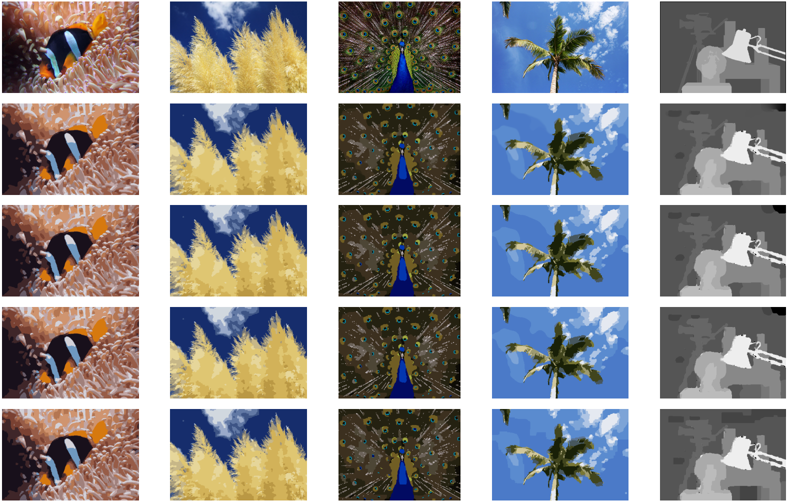




Experiments I



Experiments II



Experiments – Results

problem	1	2	3	4	5	6	7	8	9	10
# points	76800	14400	14400	129240	76800	86400	86400	76800	86400	110592
# labels	3	4	4	4	8	12	12	12	12	16
mean # iter.	7.27	7.9	8.05	10.79	31.85	49.1	49.4	49.4	49.7	66.1
a priori ε	1.	1.	1.	1.	1.	1.	1.	1.	1.	2.6332
a posteriori										
- first-max	0.0007	0.0231	0.2360	0.0030	0.0099	0.0102	0.0090	0.0101	0.0183	0.0209
- prob. best	0.0010	0.0314	0.1073	0.0045	0.0177	0.0195	0.0174	0.0219	0.0309	0.0487
- prob. mean	0.0007	0.0231	0.0547	0.0029	0.0138	0.0152	0.0134	0.0155	0.0247	0.0292

Algorithm 1 (Randomized Rounding in BV)

1. **Input:** $u^0 \in \mathrm{BV}(\Omega, \Delta_L)$
2. **For** $k = 1, 2, \ldots$
3. Sample $\gamma^k = (i^k, \alpha^k) \in \{1, \ldots, L\} \times [0,1]$ uniformly
4. $u^k \leftarrow e^{i^k} 1_{\{u^{k-1}_{i^k} > \alpha^k\}} + u^{k-1} 1_{\{u^{k-1}_{i^k} \leqslant \alpha^k\}}$
5. **Output:** Limit $\bar{u}$ of (u^k)

- Parameter space: *sequences* $\gamma \in \Gamma := (\{1, \ldots, L\} \times [0,1])^{\mathbb{N}}$

Bounds – A priori: Properties

Definition

For some sequence (γ^k), if (u_γ^k) becomes stationary at some $u_\gamma^{k'} \in \mathbb{N}$, denote output $\bar{u}_\gamma := u_\gamma^{k'}$. For some functional $f : \mathrm{BV}(\Omega)^L \to \mathbb{R}$, define

$$\begin{aligned} f(\bar{u}_\gamma): \quad & \Gamma^{\mathbb{N}} \to \mathbb{R} \cup \{+\infty\} \\ \gamma \in \Gamma^{\mathbb{N}} \quad & \mapsto f(\bar{u}_\gamma) := \begin{cases} f(u_\gamma^{k'}), & (u_\gamma^k) \text{ stationary at } u_\gamma^{k'} \in \mathrm{BV}(\Omega)^L, \\ +\infty, & \text{otherwise.} \end{cases} \qquad (*) \end{aligned}$$

- Can show: $\mathbb{P}_\gamma(f(\bar{u}_\gamma) < \infty) = 1$.
  - Generates "output" in finite time almost surely
  - Output is in $\mathrm{BV}(\Omega)^L$ almost surely

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- Truncated linear metric: $d(i,j) = \min\{3, |i-j|\}$

original

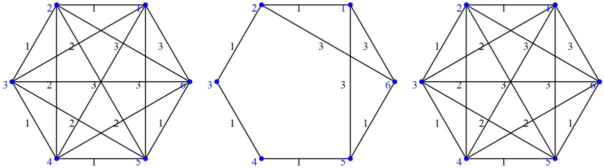
$$\begin{pmatrix} 0 & 1 & 2 & 3 & 3 & 3 \\ 1 & 0 & 1 & 2 & 3 & 3 \\ 2 & 1 & 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 & 1 & 2 \\ 3 & 3 & 2 & 1 & 0 & 1 \\ 3 & 3 & 3 & 2 & 1 & 0 \end{pmatrix}$$

reduced

$$\begin{pmatrix} 0 & 1 & \infty & \infty & 3 & 3 \\ 1 & 0 & 1 & \infty & \infty & 3 \\ \infty & 1 & 0 & 1 & \infty & \infty \\ \infty & \infty & 1 & 0 & 1 & \infty \\ 3 & \infty & \infty & 1 & 0 & 1 \\ 3 & 3 & \infty & \infty & 1 & 0 \end{pmatrix}$$

completion

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 3 & 3 \\ 1 & 0 & 1 & 2 & 3 & 3 \\ 2 & 1 & 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 & 1 & 2 \\ 3 & 3 & 2 & 1 & 0 & 1 \\ 3 & 3 & 3 & 2 & 1 & 0 \end{pmatrix}$$



15 8 15

- Half the number of constraints