Sublabel-Accurate Relaxation of Nonconvex Energies

Thomas Möllenhoff$^{1,*}$

Emanuel Laude$^{1,*}$, Michael Moeller$^1$, Jan Lellmann$^2$, Daniel Cremers$^1$

*contributed equally

$^1$TU Munich
$^2$University of Lübeck
Variational Approaches in Computer Vision

- Image denoising
- Stereo matching
- Optical flow
Variational Approaches in Computer Vision

Minimize energies of the form

$$\min_{u: \Omega \to \Gamma} \int_\Omega \rho(x, u(x)) + \lambda \cdot |\nabla u(x)| \, dx$$
Variational Approaches in Computer Vision

image denoising  stereo matching  optical flow

Minimize energies of the form

$$\min_{u: \Omega \rightarrow \Gamma} \int_{\Omega} \rho(x, u(x)) + \lambda \cdot |\nabla u(x)| \, dx$$

Challenges:

- Nonconvex data term $\rho : \Omega \times \Gamma \rightarrow \mathbb{R}$
- Continuous range $\Gamma = [\gamma_{\text{min}}, \gamma_{\text{max}}] \subset \mathbb{R}$
Discrete Multilabel Optimization [Ishikawa, TPAMI '03]

+ optimality guarantees

- discretization of $\Omega$ ⇒ grid bias
- discretization of $\Gamma$ ⇒ label bias
Continuous Lifting [Pock et al., ECCV ’08]

+ optimality guarantees
+ isotropic regularization $\Rightarrow$ no grid bias
$-$ discretization of $\Gamma$ $\Rightarrow$ label bias
Sublabel-Accurate Representation
Sublabel-Accurate Representation
Sublabel-Accurate Representation

traditional representation

2 labels, 0.07 GB
Sublabel-Accurate Representation

traditional representation

4 labels, 0.14 GB
Sublabel-Accurate Representation

traditional representation

8 labels, 0.27 GB
Sublabel-Accurate Representation

traditional representation

16 labels, 0.54 GB
Sublabel-Accurate Representation

traditional representation

32 labels, 1.09 GB
Sublabel-Accurate Representation

traditional representation

64 labels, 2.17 GB
traditional representation
128 labels, 4.34 GB
Sublabel-Accurate Representation

traditional representation
128 labels, 4.34 GB

sublabel representation
8 labels, 0.57 GB
Sublabel-Accurate Representation

traditional representation
128 labels, 4.34 GB

sublabel representation
8 labels, 0.57 GB

\[
\rho(u)
\]

\(\Gamma\)
Sublabel-Accurate Representation

traditional representation
128 labels, 4.34 GB

sublabel representation
8 labels, 0.57 GB
**Sublabel-Accurate Representation**

traditional representation
128 labels, 4.34 GB

sublabel representation
8 labels, 0.57 GB
Related Work and Contribution

MRFs with continuous state spaces / continuous graphical models
[Zach, Kohli, ECCV ’12], [Fix, Agarwal, ECCV ’14]

Key contributions of this work:

+ First spatially continuous fully sublabel-accurate formulation
+ Provably tightest local convex relaxation
+ Unification of lifting and direct convex optimization
Related Work and Contribution

MRFs with continuous state spaces / continuous graphical models
[Zach, Kohli, ECCV ’12], [Fix, Agarwal, ECCV ’14]

Partially sublabel-accurate spatially continuous multilabeling
[Lellmann et al. ICCV ’13]
Related Work and Contribution

MRFs with continuous state spaces / continuous graphical models
[Zach, Kohli, ECCV ’12], [Fix, Agarwal, ECCV ’14]

Partially sublabel-accurate spatially continuous multilabeling
[Lellmann et al. ICCV ’13]

Key contributions of this work:

+ First spatially continuous fully sublabel-accurate formulation
Related Work and Contribution

MRFs with continuous state spaces / continuous graphical models
[Zach, Kohli, ECCV ’12], [Fix, Agarwal, ECCV ’14]

Partially sublabel-accurate spatially continuous multilabeling
[Lellmann et al. ICCV ’13]

Key contributions of this work:
+ First spatially continuous fully sublabel-accurate formulation
+ Provably tightest local convex relaxation
Related Work and Contribution

MRFs with continuous state spaces / continuous graphical models
[Zach, Kohli, ECCV ’12], [Fix, Agarwal, ECCV ’14]

Partially sublabel-accurate spatially continuous multilabeling
[Lellmann et al. ICCV ’13]

Key contributions of this work:
+ First spatially continuous fully sublabel-accurate formulation
+ Provably tightest local convex relaxation
+ Unification of lifting and direct convex optimization
Traditional multilabeling methods [Ishikawa, TPAMI '03], [Pock et al., ECCV '08] specify the cost only at the labels. This leads to a linear relaxation, which is easy to optimize. However, assigning meaningful cost for solutions between the labels is a more complex task. The proposed relaxation is nonlinear but still convex!
Piecewise Linear versus Piecewise Convex Lifting

Traditional multilabeling methods [Ishikawa, TPAMI '03], [Pock et al., ECCV '08] specify the cost only at the labels. This leads to a linear relaxation, which is easy to optimize.

We assign meaningful cost for solutions between the labels. The proposed relaxation is nonlinear, but still convex!

\[ \rho(x, u) \]

\[ \rho^{**}(x, u) \]

\( u \)
Piecewise Linear versus Piecewise Convex Lifting

Traditional multilabeling methods [Ishikawa, TPAMI '03], [Pock et al., ECCV '08] specify the cost only at the labels. Leads to a linear relaxation, easy to optimize. We assign meaningful cost for solutions between the labels. The proposed relaxation is nonlinear, but still convex!
Traditional multilabeling methods [Ishikawa, TPAMI ’03], [Pock et al., ECCV ’08] specify the cost only at the labels. This leads to a linear relaxation, which is easy to optimize. However, we assign meaningful cost for solutions between the labels. The proposed relaxation is nonlinear, but still convex!
Traditional multilabeling methods [Ishikawa, TPAMI '03], [Pock et al., ECCV '08] specify the cost only at the labels. Leads to a linear relaxation, easy to optimize.

We assign meaningful cost for solutions between the labels. The proposed relaxation is nonlinear, but still convex!
Traditional multilabeling methods [Ishikawa, TPAMI '03], [Pock et al., ECCV '08] specify the cost only at the labels. Leads to a linear relaxation, easy to optimize. We assign meaningful cost for solutions between the labels. The proposed relaxation is nonlinear, but still convex!
Piecewise Linear versus Piecewise Convex Lifting

Traditional multilabeling methods [Ishikawa, TPAMI '03], [Pock et al., ECCV '08] specify the cost only at the labels. Leads to a linear relaxation, easy to optimize. We assign meaningful cost for solutions between the labels.

The proposed relaxation is nonlinear, but still convex!
Traditional multilabeling methods [Ishikawa, TPAMI '03], [Pock et al., ECCV '08] specify the cost only at the labels. Leads to a linear relaxation, easy to optimize. We assign meaningful cost for solutions between the labels. The proposed relaxation is nonlinear, but still convex!
Traditional multilabeling methods [Ishikawa, TPAMI '03], [Pock et al., ECCV '08] specify the cost only at the labels. Leads to a linear relaxation, easy to optimize. We assign meaningful cost for solutions between the labels. The proposed relaxation is nonlinear, but still convex!
Traditional multilabeling methods [Ishikawa, TPAMI ’03], [Pock et al., ECCV ’08] specify the cost only at the labels.
Traditional multilabeling methods [Ishikawa, TPAMI ’03], [Pock et al., ECCV ’08] specify the cost only at the labels.

- Leads to a linear relaxation, easy to optimize.
Traditional multilabeling methods [Ishikawa, TPAMI ’03], [Pock et al., ECCV ’08] specify the cost only at the labels.

- Leads to a linear relaxation, easy to optimize.
Traditional multilabeling methods [Ishikawa, TPAMI ’03], [Pock et al., ECCV ’08] specify the cost only at the labels. Leads to a linear relaxation, easy to optimize. We assign meaningful cost for solutions between the labels.
Traditional multilabeling methods [Ishikawa, TPAMI '03], [Pock et al., ECCV '08] specify the cost only at the labels.

- Leads to a linear relaxation, easy to optimize.
- We assign meaningful cost for solutions between the labels.
- The proposed relaxation is nonlinear, but still convex!
Tightest Convex Extension

\[ \rho^*(u) = \sup_{v \in C} \langle u, v \rangle, \quad C = \{ v \in \mathbb{R}^L | \forall i, A_i v \in \text{epi}(\rho^*_{i-1}) \} \]
Tightest Convex Extension

\[ 1_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

Proposition: The tightest convex extension is given by

\[ \rho^* (u) = \sup_{v \in C} \langle u - 1, v \rangle, \quad C = \{ v \in \mathbb{R}^L | A_i v \in \text{epi} (\rho^* i), \forall i \} \]
Tightest Convex Extension

\[ 1_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad 1_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

Proposition: The tightest convex extension is given by

\[
\rho^* (u) = \sup_{v \in C} \langle [u - 1]^T, v \rangle,
\]

where

\[
C = \{ v \in \mathbb{R}^L | A_i v \in \text{epi}(\rho^*_i), \forall i \}\]
Tightest Convex Extension

\[
1_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad 1_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad 1_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

\[
\gamma_1, \gamma_2, \gamma_3, \rho_1, \rho_2
\]

\[
\rho(u) = \begin{cases} 
\rho(\gamma_i + \alpha(\gamma_i + 1 - \gamma_i)), & \text{if } u = i - 1 + \alpha(1_i - 1_i - 1) \\
\infty, & \text{otherwise.}
\end{cases}
\]

**Proposition:** The tightest convex extension is given by

\[
\rho^* (u) = \sup_{v \in C} \langle u - 1, v \rangle,
\]

where

\[
C = \{ v \in \mathbb{R}^L : A_i v \in \text{epi}(\rho_i^*) , \forall i \}.
\]
Tightest Convex Extension

\[ 1_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad 1_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad 1_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]

Proposition: The tightest convex extension is given by

\[
\rho^\ast(u) = \sup_{v \in C} \langle u - 1, v \rangle, \quad C = \{v \in \mathbb{R}^L | A_i v \in \text{epi}(\rho_i^\ast), \forall i \}\]

\[\gamma_1 \rho_1(u) = \begin{cases} 
\rho_i(\gamma_i + \alpha(\gamma_i + 1 - \gamma_i)), & \text{if } u = i - 1 + \alpha(1 - 1) \\
\infty, & \text{otherwise.} 
\end{cases}\]
Tightest Convex Extension

\[
\begin{align*}
1_0 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
1_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
1_2 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\end{align*}
\]

\[
\gamma_1 = \rho(u) = \begin{cases} 
\rho(\gamma_i + \alpha(\gamma_i + 1 - \gamma_i)) & \text{if } u = 1 \\
\infty & \text{otherwise}
\end{cases}
\]

Proposition: The tightest convex extension is given by

\[
\rho^{**}(u) = \sup_{v \in C} \langle u - 1, v \rangle,
\]

where

\[
C = \{ v \in \mathbb{R}^L \mid A_i v \in \text{epi}(\rho^{*}_i), \forall i \}
\]
Tightest Convex Extension

\[ T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \]

\[ T_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

\[ T_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]

Proposition: The tightest convex extension is given by

\[ \rho^* (u) = \sup_{v \in C} \langle u - 1, v \rangle \]

\[ C = \{ v \in \mathbb{R}^L | A_i v \in \text{epi}(\rho_i^*) \}, \forall i \]
Tightest Convex Extension

$1_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $1_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $1_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\rho(u) = \begin{cases} \rho(\gamma_i + \alpha(\gamma_i + 1 - \gamma_i)), & \text{if } u = \gamma_i - 1 + \alpha(1_i - 1_i - 1) \\ \infty, & \text{otherwise.} \end{cases}$

**Proposition:** The tightest convex extension is given by $\rho^{**}(u) = \sup_{v \in C} \langle u - 1, v \rangle$, $C = \{v \in \mathbb{R}^L | A_i v \in \text{epi}(\rho^{*}_i), \forall i\}$. 
Tightest Convex Extension

\[ 1_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad 1_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad 1_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]

\[ \rho(u) = \begin{cases} 
\rho(\gamma_i + \alpha(\gamma_{i+1} - \gamma_i)), & \text{if } u = 1_{i-1} + \alpha(1_i - 1_{i-1}), \\
\infty, & \text{otherwise.}
\end{cases} \]
Tightest Convex Extension

\[
\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} \gamma_3 \end{bmatrix}
\]

\[\rho(u) = \begin{cases} 
\rho(\gamma_i + \alpha(\gamma_{i+1} - \gamma_i)), & \text{if } u = 1_{i-1} + \alpha(1_i - 1_{i-1}), \\
\infty, & \text{otherwise.}
\end{cases}\]

**Proposition:** The tightest convex extension is given by

\[\rho^{**}(u) = \sup_{v \in \mathcal{C}} \left\langle \begin{bmatrix} u & -1 \end{bmatrix}^T, v \right\rangle\]
Tightest Convex Extension

\[ 1_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad 1_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad 1_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]

\[
\rho(u) = \begin{cases} 
\rho(\gamma_i + \alpha(\gamma_{i+1} - \gamma_i)), & \text{if } u = 1_{i-1} + \alpha(1_i - 1_{i-1}), \\
\infty, & \text{otherwise.}
\end{cases}
\]

**Proposition:** The tightest convex extension is given by

\[
\rho^{**}(u) = \sup_{v \in \mathcal{C}} \left\langle \begin{bmatrix} u \\ -1 \end{bmatrix}^T, v \right\rangle, \quad \mathcal{C} = \{ v \in \mathbb{R}^L \mid A_i v \in \text{epi}(\rho_i^*) \}, \forall i
\]

Thomas Möllenhoff (TU Munich) Sublabel-Accurate Relaxation of Nonconvex Energies 7/1
Tightest Convex Extension

The tightest convex extension is given by

\[
\rho^{**}(u) = \sup_{v \in C} \left\langle \begin{bmatrix} u & -1 \end{bmatrix}^T, v \right\rangle, \quad C = \{ v \in \mathbb{R}^L \mid A_i v \in \text{epi}(\rho_i^*), \forall i \}
\]
**Proposition:** Tight local convex extension for lifted regularizer is

\[
\int_{\Omega} |\nabla u| \, dx \leftrightarrow \sup_{p:\Omega \to \mathcal{K}} \langle u, \text{Div } p \rangle, \quad \mathcal{K} = \{ p \mid \|p_i\| \leq \gamma_{i+1} - \gamma_i, \forall i \}
\]
Proposition: Tight local convex extension for lifted regularizer is

$$\int_{\Omega} |\nabla u| dx \leftrightarrow \sup_{p: \Omega \rightarrow \mathcal{K}} \langle u, \text{Div } p \rangle, \quad \mathcal{K} = \{ p \mid \|p_i\| \leq \gamma_{i+1} - \gamma_i, \forall i \}$$

- Leads to convex-concave saddle-point problem

$$\min_{u: \Omega \rightarrow \mathbb{R}^{L-1}} \max_{v: \Omega \rightarrow \mathcal{C}} \langle u, \text{Div } p \rangle + \left\langle \left[ \begin{array}{c} u \\ -1 \end{array} \right]^T, v \right\rangle$$
Numerical Optimization

**Proposition:** Tight local convex extension for lifted regularizer is

\[
\int_\Omega |\nabla u| \, dx \leftrightarrow \sup_{p:\Omega \to \mathcal{K}} \langle u, \text{Div} \, p \rangle, \quad \mathcal{K} = \{ p \mid \|p_i\| \leq \gamma_{i+1} - \gamma_i, \forall i \}
\]

- Leads to convex-concave saddle-point problem

\[
\min_{u:\Omega \to \mathbb{R}^{L-1}} \max_{v:\Omega \to \mathbb{C}} \langle u, \text{Div} \, p \rangle + \left\langle \left[ u \quad -1 \right]^T, v \right\rangle
\]

- Solved on GPU using a first-order primal-dual algorithm

[Pock, Cremers, Bischof, Chambolle, ICCV '09]
Backprojecting the Lifted Solution
Backprojecting the Lifted Solution
Backprojecting the Lifted Solution
Backprojecting the Lifted Solution
Backprojecting the Lifted Solution
Backprojecting the Lifted Solution
Backprojecting the Lifted Solution

\[ u^* \]
Convex Case: \( \rho (x, u(x)) = (u(x) - f(x))^2 \)

direct, no labels
0.6s, 11.78 MB
Convex Case: $\rho(x, u(x)) = (u(x) - f(x))^2$
Convex Case: \( \rho(x, u(x)) = (u(x) - f(x))^2 \)
Convex Case: \( \rho(x, u(x)) = (u(x) - f(x))^2 \)
Convex Case: $\rho(x, u(x)) = (u(x) - f(x))^2$
Convex Case: \( \rho(x, u(x)) = (u(x) - f(x))^2 \)
Stereo Matching, $\rho(x, u(x)) = \| I_1(x) - I_2(x_1 + u(x), x_2) \|$
Stereo Matching,

\[ \rho(x, u(x)) = \| I_1(x) - I_2(x_1 + u(x), x_2) \| \]

traditional, 2 labels

sublabel, 2 labels
Stereo Matching, \( \rho(x, u(x)) = \| I_1(x) - I_2(x_1 + u(x), x_2) \| \)
Stereo Matching, \( \rho(x, u(x)) = \| I_1(x) - I_2(x_1 + u(x), x_2) \| \)
Stereo Matching, $\rho(x, u(x)) = \|I_1(x) - I_2(x_1 + u(x), x_2)\|$

diagram showing traditional and sublabel matching with 2, 4, 8, and 16 labels.
Conclusion

- We proposed a sublabel-accurate relaxation for a certain class of nonconvex energies

https://github.com/tum-vision/sublabel_relax
Conclusion

- We proposed a sublabel-accurate relaxation for a certain class of nonconvex energies
- Requires far fewer labels than traditional lifting techniques

https://github.com/tum-vision/sublabel_relax
Conclusion

- We proposed a sublabel-accurate relaxation for a certain class of nonconvex energies
- Requires far fewer labels than traditional lifting techniques
- Leads to substantial improvements in runtime and memory
Conclusion

- We proposed a sublabel-accurate relaxation for a certain class of nonconvex energies
- Requires far fewer labels than traditional lifting techniques
- Leads to substantial improvements in runtime and memory
- It generalizes traditional lifting methods from piecewise linear to piecewise convex approximations
Conclusion

- We proposed a sublabel-accurate relaxation for a certain class of nonconvex energies
- Requires far fewer labels than traditional lifting techniques
- Leads to substantial improvements in runtime and memory
- It generalizes traditional lifting methods from piecewise linear to piecewise convex approximations

https://github.com/tum-vision/sublabel_relax
Comparison with [Zach, Kohli, ECCV ’12]

- Denoising with robust data term

\[ \rho(x, u) = (\alpha/2) \min \{ \nu, (u - f(x))^2 \} \]

- Special case of our method: anisotropic regularizer \( \| \nabla u \|_1 \)
- Our relaxation uses only **half** the number of variables

proposed tight relaxation, 33 labels, **Energy:** 194836

[Zach, Kohli, ECCV ’12], DC-MRF, 33 labels, Energy: 194845